A CRITICAL STUDY OF THE THEORY AND DEVELOPMENT OF METHODS OF APPLICATION OF THE OPEN MOVING COIL GALVANOMETER

A THESIS

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INTRODUCTION

The present form of the moving coil galvanometer has resulted from a large amount of work, both theoretical and experimental, which has been done upon the instrument since its introduction about thirty years ago. These various researches, aside from tests and comparisons made with and between existing instruments, may be placed in four groups, according to purpose, as follows:

a) To determine the design best suited to each set of conditions under which an instrument is to be used.

b) To increase the sensitivity , both of deflection and ballistic galvanometers.

c) To determine the errors resulting from the characteristic relations ~~of~~ between parts of the instrument, with methods for overcoming or minimizing these errors.

d) To extend the uses of the instrument.

The work with which this paper deals was taken up with the third and fourth of these objects in view. The first step was to make a careful study of the theory of this type of galvanometer by reviewing the papers which have been published upon the subject. From this point the development of methods was carried forward to fill in such gaps as were found still to exist.

For purposes of theoretical treatment, an ideal galvanometer will be assumed, in which certain conditions may be regarded as fulfilled. These conditions, although not exactly in agreement with fact, are such good approximations as to warrant their assumptions; this is shown in the excellent agreement between theory and experiment, as will be shown in this paper.

The conditions above mentioned are: A rectangular coil without auxiliary damping devices, suspended by means of perfectly elastic fibers in such a way that it may move freely about a vertical axis which is at the same time the vertical axis of symmetry of the magnetic parts of the instrument. The magnetic field is supposed perfectly radial, the coil cutting the lines of force perpendicularly, regardless of angular position; this is effected by means of a soft iron cylindrical core, placed coaxially within the coil but not in contact with the latter. The self inductance of the coil is assumed of such value that there is no lag of current behind the electromotive force producing it. In every case the damping moment will be considered proportional to the angular velocity of the coil. This is strictly true under the foregoing assumptions for that part of the damping moment arising from electromagnetic induction, and is closely in accordance with fact as regards air damping. Hence, the greater the ratio of electromagnetic to air damping, the more closely is the last mentioned assumption realized.

The work of this paper will be presented in eight sections, the general plan in each of which is, first, to give the theoretical treatment of the subject matter of that section, resulting in certain equations. When these equations are practically applicable, a discussion will be given as to the procedure in applying the formulae, with the experimental precautions to be observed. Finally, the description of apparatus used in testing the equations, together with the results of the tests, will complete the section.

In the list which follows are given the symbols which will be used to designate the quantities used in the general discussions. For particular cases, other symbols will be introduced, and the general symbols will be provided with subscripts, all of which will be explained in connection with the equations in which they first appear.

GENERAL NOTATION

θ = angular displacement at time t.

θ' = first time derivative of θ, or angular velocity.

θ'' = second time derivative of θ, or angular acceleration.

I_o = moment of inertia of coil.

$2f$ = proportionality constant between damping moment and angular velocity of coil.

q^2 = elastic torque constant of suspensions, or torque per unit angle.

$a = \frac{f}{I_o}$; $b = \sqrt{\frac{q^2}{I_o} - \frac{f^2}{I_o^2}}$; $\beta = \sqrt{\frac{f^2}{I_o^2} - \frac{q^2}{I_o}}$.

n = number of turns on coil.

A = area of coil.

H = average field intensity.

M = nAH.

ρ = ratio of any elongation of coil to the one next following, in damped periodic motion.

λ = logarithmic decrement on open circuit.

Λ = logarithmic decrement (general).

T = period of coil.

i = current.

Q = quantity of electricity.

e = electromotive force.

r = resistance.

K = ballistic constant.

k = current constant.

ϕ = angle of steady deflection produced by current i.

$$h(\Lambda) = \sqrt{1 + \frac{\Lambda^2}{\pi^2}}.$$

$$\Lambda' = \frac{\Lambda}{H(\Lambda)}.$$

SECTION I. GENERAL THEORY.

Having in mind a galvanometer coil according to the specifications given on page 2, our first problem, on which all subsequent work is to be based, is to determine the differential equations of motion of such a coil under practical conditions. Next, the solutions of these equations are to be found, and into them are to be introduced such special conditions as arise out of practice, in order to evaluate the integration constants.

The differential equations which we find are of a common type, and their solution presents no difficulty. They have been used repeatedly in discussions involving motion resulting from a force proportional to a displacement, with a resistance proportional to the velocity; they have likewise been used in the treatment of alternating and oscillating current phenomena. This section, therefore, will present nothing which is entirely original; the subject matter given is, however, fundamental, and is ~~given~~ included for the sake of completeness.

The most general representation of the motion of the galvanometer coil is of the form

$$I_0\theta'' + 2f\theta' + q^2\theta = f(t), \tag{1}$$

the right-hand member of the equation reducing to zero or a constant in the cases which will be considered in the present work. When $f(t) = 0$, the motion will be designated as "free", in contradistinction to the motion resulting when a constant torque, $F(t) = \text{const.}$, is applied to the moving system. In the above equation the damping coefficient $2f$ may be regarded as being composed of two parts: we may write

$$f_0 + f_1 = f, \tag{2}$$

where f_0 is the part of f which obtains when the coil is swinging on open circuit, and where f_1 is the additional part due to the generator action of the coil when it is part of a closed circuit. In the former case, f_1 is, of course, zero. Thus, with the designation given, f_0 takes account of the air-damping and that due to eddy-currents. All current damping, whether it be due to eddy currents or to currents established in the circuit of which the coil is a part as a result of the swinging of the coil, is accurately proportional to the angular velocity of the coil; and air-damping is very closely so.[1]

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[1]O.M.Stewart, Phys.Rev., O.S., 16, 161, 1903; W.P. White, Phys.Rev., O.S., 19, 306, 1904; Klopsteg, Phys.Rev., N.S., 2, 391, 1913.

A. Equations of Motion of the "Free" Coil.--When $f(t) = 0$, one form of solution of equation (1) is

$$\theta = \epsilon^{-at}(C_1\epsilon^{\beta t} + C_2\epsilon^{-\beta t}), \tag{3}$$

C_1 and C_2 being the constants of integration. Three particular cases are possible, corresponding to the three conditions, respectively:

1) $I_o q^2 > f^2$; 2) $I_o q^2 = f^2$; 3) $I_o q^2 < f^2$.

Each of the cases will be treated separately; but the common initial conditions will be imposed that, at the instant $t = o$, $\theta = 0$, and $\theta' = MQ/I_o$, the quantity Q of electricity being supposed to be completely discharged through the coil im an infinitesimal interval of time.

1) Damped Periodic Motion.--When condition 1) and the initial conditions mentioned are used to evaluate the integration constants, equation (3) becomes

$$\theta = \epsilon^{-at}(MQ/I_o)(\frac{1}{b}\sin bt), \tag{4}$$

from which the angular velocity

$$\theta' = \epsilon^{-at}(MQ/I_o)(\cos bt - \frac{a}{b}\sin bt). \tag{5}$$

The time t_1 required by the coil to reach its first elongation α_1, found by solving for t in (5) when $\theta' = 0$, is

$$t_1 = \frac{1}{b} \tan^{-1} \frac{b}{a}; \tag{6}$$

subsequent consecutive maxima occur at πt_1, $2\pi t_1$, ..., the interval between two successive maxima being, therefore, π/b. Hence the period of the motion is

$$T = 2\pi/b. \tag{7}$$

This result is also deducible from the condition that θ in equation (4) be zero. In this case the period is twice the time interval between two successive passages of the coil through its zero position.

If the values for t corresponding to the successive elongations α_1, α_2, α_3, ..., be substituted in (4), the values of these respective elongations are obtained; and it is found that the ratio

$$\frac{\alpha_n}{\alpha_{n+1}} = \epsilon^{(a/b)\pi}, \tag{8}$$

a constant, which we may designate by ρ. The natural logarithm of ρ, which is $a\pi/b$, is called the logarithmic decrement of the motion; i.e.,

$$\Lambda = \log \rho = (a/b)\pi. \tag{9}$$

Equations (7) and (9) give the useful relationship

$$T = \frac{2\Lambda}{a} . \tag{10}$$

Substitution in equation (9) of the values of a and b before given results in the relation

$$\Lambda = \frac{\pi}{\sqrt{\frac{q^2 I_0}{f^2} - 1}} . \tag{11}$$

If, in equation (7), f is placed equal to zero, we obtain

$$T_0 = 2\pi \sqrt{\frac{I_0}{q^2}} , \tag{12}$$

T_0 being the period of the undamped motion of the coil. Combining this with equation (7), and making use of equation (11) in simplifying the result, we obtain

$$T = T_0 \sqrt{1 + \frac{\Lambda^2}{\pi^2}} = T_0 h(\Lambda). \tag{13}$$

2) Critically Damped Motion.--This type of motion of the galvanometer coil has considerable practical importance, and has for this reason been treated by various workers. Diesselhorst[1] points out that at this boundary condition of damping the time of return of the coil from a given deflection to within 1/n of that deflection from the zero is shorter than for any other condition

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[1]Diesselhorst, H., Ann.d.Phys., 9, 458, 1902.

of damping. He further shows that for a given instrument the maximum ballistic sensitivity occurs when ρ has a value of about 8, but that only 5 per cent. of the sensitivity is sacrificed when damping is made critical. It is held by this author that accuracy of ballistic readings is not secured unless the time of throw of the coil is at least 5 seconds; and he concludes from his deductions that, in order to combine in a given instrument the three desirable qualities of long time of throw, short time of return and high sensitivity, this instrument should have a period of about 30 seconds, and that it should be critically damped. He emphasizes the importance of having damping as nearly critical as it is possible to make it, showing by means of a curve that for small differences in damping in either direction from the critical value the time of return to within a negligible fraction of the original deflection rapidly increases.

It is the writer's experience that ballistic throws, the time of which is less than two seconds, are still determinable with accuracy; and in his opinion a galvanometer of 12 seconds' period is a more useful instrument, other things being equal, than one of 30 seconds' period, on account of the saving of time in experiments, which may be effected by the former. In a discussion of a paper read before the American Institute of Electrical Engineers, Sheldon[1] mentions using a Weston round pattern voltmeter with the resistance cut out as a ballistic galvanometer for determining hysteresis curves, and Stewart[2] and Ganz[3] give results showing the proportionality of throws obtained from Weston instruments with quantities of electricity discharged through them. Stewart, in the paper cited, shows in a theoretical treatment that in the case of critical damping throws and quantities producing them; this can be shown more directly[4], not only for the case of critical damping, but for slightly damped and overdamped motions as well.

For the case of critical damping, then, corresponding to the condition $I_0q^2 = f^2$, we have

$$\theta = (MQ/I_0)t\epsilon^{-at}, \tag{14}$$

and the angular velocity at the instant t is

$$\theta' = (MQ/I_0)(1 - at)\epsilon^{-at}. \tag{15}$$

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[1]Proc. Am. Inst. E. E., 17, 15, 1900.
[2]Stewart, O. M., Phys. Rev., 16, 158, 1903.
[3]Ganz, A. F., Science, Mar. 3, 1905.
[4]Klopsteg, M.A. Thesis, Minn., 25, 1913.

From equation (15) we find the time of throw to be

$$t_1 = 1/a = T_0/2\pi. \tag{16}$$

3) Overdamped Motion.--This case, corresponding to the condition that $I_0q^2 < f^2$ has little application in galvanometry. The equations are

$$\theta = (MQ/I_0)\epsilon^{-at}(1/2\beta)(\epsilon^{\beta t} - \epsilon^{-\beta t}), \tag{17}$$

which may also be written

$$\theta = (MQ/I_0)\epsilon^{-at}(\frac{1}{\beta} \sinh \beta t). \tag{18}$$

The angular velocity at the instant t is

$$\theta' = (MQ/I_0)\epsilon^{-at}(\cosh \beta t - \frac{a}{\beta} \sinh \beta t), \tag{19}$$

and the instant at which it becomes zero is

$$t_1 = 1/\beta \ \tanh^{-1} \beta/a. \tag{20}$$

A comparison of the last three equations with equations (4), (5) and (6), respectively, shows their forms to be similar, the only differences being that in the equations for damped periodic motion the circular functions are replaced by the corresponding hyperbolic functions, and b is replaced by β.

B. Equations of Motion of the Coil under the Influence of a Torque Arising from a Steady Current.--The differential equation representing the relation between the various torques acting upon the coil at any instant is obtained by putting the right-hand member of equation (1) equal to Mi, the steady turning moment applied to the coil when the constant current i traverses its turns. Again, as in the case of the free coil, three solutions of the differential equation are possible, corresponding to the same three conditions enumerated on page 8. Only the first two of these cases will be treated, namely the conditions of damped periodic and critically damped motions; and in each of these cases the initial conditions will be introduced that at the instant $t = 0$, $\theta = 0$ and $\theta' = 0$, to evaluate the constants of integration.

1) $I_0q^2 > f^2$: Damped Periodic Motion.-- The equations for angular position and angular velocity, respectively, at the instant t, are

$$\theta = (Mi/q^2)\left[1 - \epsilon^{-at}(\cos bt + \frac{a}{b}\sin bt)\right], \quad (21)$$

and

$$\theta' = (Mi/q^2)\epsilon^{-at}(a^2/b + b)\sin bt. \quad (22)$$

The motion described by equation (21) has a period and logarithmic decrement given by equations (7) and (9), respectively, the same as in the case of the free coil. As t increases without limit, θ assumes the value Mi/q^2.

2) $I_0 q^2 = f^2$: Critically Damped Motion.--In steady deflection work with the galvanometer this case has importance because of the convenience of manipulation; in fact, it is practically the only satisfactory condition of damping for a galvanometer to be used as a deflection instrument. We have, for this condition,

$$\theta = (Mi/q^2)(1 - \epsilon^{-at}), \tag{23}$$

and

$$\theta' = (Mi/q^2)\epsilon^{-at}. \tag{24}$$

SECTION II. RELATIONS BETWEEN DAMPING AND RESISTANCE.

A. Derivation from Periodic Motion.--Suppose the galvanometer coil to be part of a circuit the total resistance of which is r_1, and that the coil is started swinging. The value of f_1 in equation (2) may then be expressed as $M^2/2r_1$. Using equation (7) to express the values of f and f_o in terms of the corresponding logarithmic decrements on closed and open circuit, respectively, equation (2) may be written

$$2\lambda I_o/T + M^2/2r_1 = 2\Lambda_1 I_o/T_1, \tag{25}$$

where T represents the period on open circuit, corresponding to the logarithmic decrement λ, and T_1 is the period on closed circuit of resistance r_1, to which corresponds the logarithmic decrement Λ_1. Equation (13) enables us to express (25) as

$$M^2T_o/4I_o = (\Lambda_1' - \lambda')r_1, \tag{26}$$

from which we see that the resistance in the circuit and the difference between the "modified" logarithmic decrements (see general notation) on closed and open circuit, respectively, are inversely proportional. Consequently, with a given coil, we have

$$r_1/r_2 = (\Lambda_2' - \lambda')/(\Lambda_1' - \lambda'). \tag{27}$$

Theoretically, we should be able, with the aid of equation (27), to measure resistances of any value whatsoever above that value at which damping becomes critical, by a comparison method. A formula has been given[1] for the measurement of high resistances by a method of damping, in which a moving coil galvanometer is used. The formula differs from that of equation (27) in details only. It is to be noticed that if, in equation (27) all the Λ's are small, the primes may be dropped, in which case the formula given by Starling results.

The galvanometer coil being a part of the circuit, the resistance of the instrument is involved in the r's occurring in formula (27). This might suggest replacing r_1 by $r_1'+G$, and r_2 by $r_2'+G$, and solving for G, the galvanometer resistance; the primed r's represent external resistances. With this modification the formula might be used for obtaining the galvanometer resistance in terms of two known resistances, and the corresponding logarithmic decrements. The objection to the method is that the right-hand member of such a formula solved for G is expressed as the differences between small quantities, leading to a large probable error in the result. This method, employing an experimental curve instead of the equation, however, is given by Zeleny[2]. His value of

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[1] Starling, Electricity and Magnetism, p. 260.
[2] Zeleny, A., Phys. Rev., O.S., 23, 420, 1906.

a/b may be shown to be identically that of r_1/r_2 of equation (27).

B. Derivation from Critically Damped Motion.--
The condition for critical damping yields another substitution which may be made in equation (2). Remembering that now $I_0 q^2 = f^2$, and that, from equation (12) this is equal to $2\pi I_0/T_0$, equation (2) becomes

$$2\lambda I_0/T + M^2/2r_c = 2\pi I_0/T_0, \tag{28}$$

which, upon simplification in a manner similar to that used in connection with equation (27), becomes

$$M^2 T_0/4I_0 = r_c(\pi - \lambda'), \tag{29}$$

where r_c represents the total resistance in the circuit for which damping becomes critical. Analogous to (27) there results

$$r_c/r_1 = (\Lambda_1' - \lambda')/(\pi - \lambda'). \tag{30}$$

This equation, by the use of the modification above suggested, would appear, theoretically, to adapt itself better to the determination of the galvanometer resistance than a method in which all the logarithmic decrements are small. In the present instance we should be using comparison resistances of the same order of magnitude as the galvanometer resistance, thus making the probable error in the measurement smaller. This is indeed the case; but experiment shows another source of error, not so great as the other, perhaps, but still too large to make this a method of precision measurement of galvanometer resistance. It will be noticed that, in this manipulation, one of the quantities to be determined would be the external resistance in series with the galvanometer to render damping exactly critical. Experience indicates that, when damping is nearly critical, it is invariably overestimated, which means that in a direct determination of external resistance for critical damping by direct observation of the motion of the coil, the value of the resistance is invariably made considerably larger than it should be; in the writer's experience this excess averages apout 7 per cent.

There is, however, an application of equation (30) which is of great utility; it is suggested by what has been said in the preceding paragraph in regard to one's inability to judge when critical damping obtains. The application is to the determination of the external resistance which will render the motion of the coil just aperiodic. In certain experiments it is imperative to have the resistance accurately adjusted to produce this boundary condition of damping. Wenner[1] describes a

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[1]Wenner, F., Phys. Rev., O.S., 22, 192, 1906.

graphical method to accomplish the purpose which, in the writer's experience is accurate to about 5 per cent. at best, but which is useful when exactly critically damped motion is not necessarily a desideratum. Jaeger[1] gives a formula for finding the critical damping resistance in terms of logarithmic decrements and periods which is reducible to the form of equation (30), but which, in the form given, is rather cumbersome in its application.

In a later section of this paper an application of equation (27) will be made, in connection with the determination of ballistic constants on closed circuit; advantage is taken of the possibility of finding, by means of this equation, a logarithmic decrement corresponding to some predetermined resistance. Such a computation is much simplified by the use of tables giving values of $h(\Lambda)$, permitting the value of Λ' to be easily determined.[2]

C. Derivation from Overdamped Motion.-- This relation, based on equation (17), is the development

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[1]Jaeger, G., ZS.f.Instrk., 23, 261, 1903.
[2]Kohlrausch, 12th ed., p. 726.

of a suggestion in a paper by Jones[1] indicating a simple relationship between resistance in a galvanometer circuit of which the coil is overdamped and time of change of angular position of the coil by a certain amount.

For large values of t, in particular for the value, let us say t_o, equation (17) may be written with sufficient accuracy

$$\theta_1 = (MQ/2\beta I_o)\epsilon^{-(a-\beta)t_o};$$

and at a still later instant, say $t_o + \tau$,

$$\theta_2 = (MQ/2\beta I_o)\epsilon^{-(a-\beta)(t_o+\tau)}.$$

The ratio of θ_2 to θ_1 is

$$R = \epsilon^{(a-\beta)\tau},$$

or, rewriting the equation in logarithmic form, expanding β, and solving for τ, (31)

$$\tau = (\log R)/(q^2/2aI_o).$$

Here we have neglected higher powers of $q^2/2aI_o$, or $q^2/2f$, since, in overdamped motion f is large in comparison with q^2. Remembering that in this case that part of f which is due to the generator action of the coil is large in comparison with f_0, the part which obtains when the coil swings on open circuit, and remembering that for this reason r, the resistance in the circuit is inversely proportional to f, we deduce from (31) that

$$r\tau = \text{constant}. \tag{32}$$

Let r be composed of the external resistance r' and the galvanometer resistance G. Then, from (32), if r' in the first case is zero,

$$G/(r'+G) = \tau_2/\tau_1;$$

or,

$$G = \tau_2 r'/(\tau_1-\tau_2). \tag{33}$$

This equation should be directly applicable to the determination of galvanometer resistance to a fair degree of approximation. From the development it is seen that τ_1 is the time of "creep" of the overdamped coil from some angle θ_1 to some smaller angle--any angle whatever--say θ_2, when the galvanometer is short-circuited; and that τ_2 is the time of creep over the same angle when the external resistance r' is inserted in the circuit with the coil. The conditions to be fulfilled are complied with in the ordinary ballistic galvanometer: rather weak control, and small damping on open circuit.

D. Experimental.--To subject equations (27) and (30) to experimental test, observations were taken upon galvanometers in connection with work which will be mentioned in Secs. 5 and 8 of this paper. For the present the statement may be made that the results show the equations to hold for the ordinary galvanometer with an accuracy of about 0.2 per cent.

Equation (33) was tested with a Leeds and Northrup type P galvanometer, a resistance box and a stop watch. The resistance of the galvanometer was first measured with a Wheatstone bridge, and found to be 128.7 ohms. The time interval determined in each of the cases tabulated below was that required, after a throw of 20 cm., by the coil to "creep" from 15 cm. down to 1.5 cm. In the third column of the table are given the results obtained from sets 2 and 1, 3 and 1, 4 and 1, respectively, when used in equation (33).

Obs. No.	r'	T	G
1	0	29.71	[illegible]
2	50.0	21.50	130.9
3	100.0	16.82	130.6
4	150.0	13.84	130.9

SECTION III. APPLICATION OF THE DAMPED COIL TO THE MEASUREMENT OF MAGNETIC FIELDS.

I. THEORETICAL.

A. Derivation from Periodic Motion.--From equation (26) we may write directly

$$H = C\sqrt{\frac{r_1}{T_o}(\Lambda_1{}'-\lambda')}, \tag{34}$$

where C represents the constant for any particular coil, $\sqrt{4x10^9 I_o/nA}$. In the above equation r_1 is expressed in ohms. T_o, the period of the undamped coil is, for any particular value of the magnetic field, a constant; and if it were not for the magnetic impurities[1] in the coil, which are invariably present, this factor might be included in C. The equation was developed for radial fields; but so long as the angle of vibration of the coil is smaller than that which corresponds to a deflection of 10 cm. on a scale at 50 cm. from the mirror, the error in applying the formula to parallel fields is less than 0.5 per cent., theoretically.

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[1]Zeleny, A., Phys.Rev., O.S., 32, 297, 1911.

B. Derivation from Conditions for Critical Damping.--The equation for this case follows directly from equation (29), the form being similar to that of (34). Solving (29) for H,

$$H = C\sqrt{\frac{r_o}{T_o}(\pi - \lambda')}, \tag{35}$$

where C has the same value as in the preceding equation.

Without going into this case further, it may be stated that equation (35) is not suited to accurate measurements of field intensity because it involves a determination of r_o, the resistance in the galvanometer coil circuit at which damping is exactly critical. The difficulty of doing this directly was pointed out on page 19 of this paper.

II. EXPERIMENTAL.

The constant C which enters into equation (34) may be found by direct observation of the quantities involved. Another method which suggests itself is to apply (34) to the measurement of a known magnetic field, and to find C from these observations; this would reduce the measurement by means of the equation to one of comparison.

To test the accuracy of the method of equation (34) the values obtained by its use for a series of magnetic field intensities were compared with the results of determinations of the same intensities by the ballistic method.

For the purpose of obtaining a variable field of known intensity, an electromagnet with large plane pole pieces was used. The field intensity corresponding to any particular value of the magnetizing current was first measured by the ballistic method with a test coil of 10 sq. cm. area and having 30 turns. The coil was placed, in its mounting, between the poles of the electromagnet, and was, by means of a spring, snapped through a half revolution when a release was operated. Throws from the test coil were compared with throws obtained from a standard mutual inductance coil, the coefficient of which had been accurately determined. The primary current in the comparison coil was measured by means of a standard resistance and potentiometer. In order to obtain a definite strength of field with a given current the magnet was, at the beginning of any determination, subjected to several cycles of hysteresis, after which the current was increased. The demagnetizing current necessary to overcome the coercive force of the iron was accurately determined, so that zero field could be obtained at will.

The magnetization curve was carefully drawn to a large scale, and from it were taken the values of the field intensity with which the results of the damped coil method were compared.

For the coil with which the measurements were made according to the formula, a Leeds and Northrup galvanometer coil of 121,7 ohms resistance was used. This was suspended in a brass case with a glass front by means of the usual phosphor bronze suspensions, using for the upper suspension a 1.5-mil strip. The coil had 395 turns, with mean dimensions 4.85 x 1.69 cm., and with a moment of inertia of 3.2o c.g.s. units. The value of the factor C was 34.80.

The values obtained by the two methods for field intensity of different magnetic fields ranging up to 1500 gausses are given in the last two columns of Table 1. The agreement, considering the fact that the ballistic measurements may be in error by about 0.5 per cent., is seen to be good; for most of the values it is within one percent.

(To follow page 27)

					Field Intensity	
Magn. Current	T_o	λ'	r_1	Λ_1'	Eq. (34)	Ballist. Mthd.
0.0998	7.948	.01102	121.7	.01170	3.40	4.4
0.1199	7.945	.01101	121.7	.02242	14.54	16.5
0.1885	7.967	.01128	121.7	.1872	57.0	57.0
0.263	7.974	.01190	621.7	.1277	104.5	105.7
0.332	7.948	.01278	1,122	.1399	147.4	148.5
0.414	7.890	.01413	1,622	.1747	199.8	199.0
0.566	7.478	.01742	3,500	.1708	294.7	295
0.983	6.913	.03060	10,000	.2026	548.5	542
1.421	6.428	.04614	20,000	.2079	780	770
1.854	6.128	.06196	40,000	.1800	966	964
2.350	5.891	.07769	50,000	.2025	1,133	1,143
3.500	5.630	.09800	100,000	.1892	1,400	1,405
4.330	5.503	.10678	100,000	.2129	1,528	1,503

Table 1.

SECTION IV. CALCULATION OF WIRE SIZE FOR DAMPING RECTANGLE.

It has been pointed out on page 19 that when a galvanometer is used for deflection work, the resistance in the circuit may be so adjusted as to make damping critical. It is desirable, in fact almost imperative, to have this condition of damping, but it is not always expedient to shunt the coil with a resistance, for this may greatly diminish the sensitivity of the instrument. The alternative is to attach to the coil a rectangle of copper wire of such size that, with the conditions which obtain in the circuit, the additional damping due to the rectangle shall render the motion of the coil just aperiodic. In this way no sensitivity is sacrificed in bringing about the boundary condition of damping. When the resistance in the galvanometer circuit is high, many times as great as that of the galvanometer resistance, the logarithmic decrement will be practically the same as that on open circuit, and the damping rectangle to produce critical damping on open circuit may be employed. When, however, the resistance is diminished, so as to increase appreciably the logarithmic decrement, the wire size of the rectangle to be selected must be smaller than before, since less additional damping is required. In the following discussion a method is to be outlined which will enable one to select a rectangle of that particular wire size which will bring about the desired result under the conditions of the experiment.

We shall, in the first place, suppose the number of turns on the coil to be given; this number is obtainable from the manufacturers, who keep a record of the data on each coil manufactured by them. Let it be assumed that a single closed turn of the same mean area and of the same moment of inertia as the coil replaces the latter, and that the resistance of this single turn is r_0. Then we have, by equation (29),

$$r_0 = \frac{A^2H^2T_0}{4\times10^9 I_0(\pi - \Lambda_0')}, \tag{36}$$

where r_0 is expressed in ohms, and Λ_0 represents the logarithmic decrement of the coil in the circuit in which it is to be used. If D is the diameter of the wire to be used, S its specific resistance, l and w the length and width, respectively, of the rectangle --this, in general, will be the length and width, respectively, of the coil--we may also write

$$r_o = \frac{8S(1+w)}{\pi D^2} . \tag{37}$$

Combining the last two equations, and making use of the relation expressed in equation (26), we find

$$D = n\sqrt{\frac{8S(1+w)(\pi-\Lambda_o')}{r_1\pi(\Lambda_1'-\lambda')}} . \tag{38}$$

When Λ_o' is small in comparison with π, which is the case with some galvanometers when the resistance in the circuit is very high, and which also holds true in many ballistic instruments, equation (38) simplifies to

$$D = n\sqrt{\frac{8S(1+w)}{r_1(\Lambda_1'-\lambda')}} . \tag{39}$$

Discussion and Experimental Test.--The rectangle for which the wire size is given by equations (38) or (39), when attached to the forward edge of the coil, is in a weaker magnetic field than is the mean turn of the coil, since it is displaced forward from that position by several millimeters. In a number of instruments the ratio of the field intensities at the two positions mentioned was determined, and the average value found was 1.30. This, of course, ought to be determined for each individual instrument, if the boundary condition of damping is to be attained with exactness. The value of D found should be multiplied by this ratio.

It should be observed that the addition of the mass of the rectangle to that of the coil has the effect of slightly increasing I_0. The effect of this added mass, assuming it to be of such kind as not to influence damping, would be to diminish both A_1 and λ to a slight degree. The difference between the two quantities enters under the radical sign, indicating that the error from this source should be small. Practically, one is also limited in his selection to the standard wire sizes, which admits of some variation in D.

To determine the effect of the increased mass of the coil, brought about by the addition of the rectangle, an experiment was tried in which an "open" rectangle of the wire size given by the formula was attached to the coil. A redetermination of D gave a slightly different value, not sufficiently different, however, to necessitate using a different wire gauge.

SECTION V. DERIVATION OF EQUATIONS FOR BALLISTIC CONSTANTS.

A. Theoretical.

It has been shown[1] that whatever the condition of damping in a moving coil galvanometer, the throws of the coil are proportional to the quantity of electricity producing them. The proportionality factor, or ballistic constant, which is the factor by which the throws are to be multiplied to give the quantity, may be found directly with accuracy by discharging a quantity from some standard device through the instrument and noting the throw. The theory developed in Section I of this paper yields a number of equations for the ballistic constant in terms of quantities easily determinable, from which the factor is accurately obtainable; this obviates the necessity of using standard devices, which are not always at hand. The equations mentioned will be developed in this section. Of these equations, (43), (44) and (51) are not original with the author; especially equation (44) is developed in most manuals

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[1]Klopsteg, M.A.Thesis, Minn., 1913.

of electrical measurements. They are here included for the sake of completeness.

1. Damped Periodic Motion.--Substitution of equation (6) in equation (4) gives the value α_1 of the first elongation of the coil resulting from the instantaneous discharge of the quantity of electricity Q. Thus,

$$\alpha_1 = \frac{M}{\sqrt{I_o q^2}} \epsilon^{-\frac{a}{b} \tan^{-1} \frac{b}{a}} \cdot Q, \tag{40}$$

showing, from the definition previously given, that the ballistic constant is, for electromagnetic units of quantity per radian,

$$K' = \frac{\sqrt{I_o q^2}}{M} \epsilon^{\frac{a}{b} \tan^{-1} \frac{b}{a}}. \tag{41}$$

This is the general formula, the factors in which are not readily found experimentally; the equation must therefore be put into practicable form.

In the first place, reference to equation (9) shows that the exponential in equation (41) is identically $\rho^{\frac{1}{\pi} \tan^{-1} \frac{\pi}{\lambda}}$; this is the correction for damping, and holds true for all values of the logarithmic decrement.[1] Consequently the different methods of obtaining the value of the ballistic constant are essentially methods for the experimental determination of $\sqrt{I_o q^2}/M$.

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[1]Klopsteg, Phys. Rev., N. S., 7, , 1916.

a) First Method.--When the coil, carrying the current i, is in equilibrium at the angular position ϕ, $Mi = q^2\phi$; hence the current constant k, which we shall suppose to be expressed in amperes per centimeter, is equal to q^2/M. Applying this relation and equation (13) to (41), we obtain

$$\frac{\sqrt{I_0 q^2}}{M} = \frac{Tk}{2\pi h(\Lambda)}\ , \tag{42}$$

with the resulting formula for the ballistic constant in coulombs per centimeter,

$$K = \frac{Tk}{2\pi h(\Lambda)}\ \rho^{\frac{1}{\pi}\tan^{-1}\frac{\pi}{\Lambda}}\ , \tag{43}$$

or, for slight damping,

$$K = \frac{Tk\sqrt{\rho}}{2\pi}. \tag{44}$$

Tables for the evaluation of the damping factor in (43) and of $h(\Lambda)$ for particular values of Λ are given in the twelfth edition of Kohlrausch, Lehrbuch der praktischen Physik, page 723.

b) Second Method.--Taking from equation (12) the value of $\sqrt{I_0 q^2}$ in terms of moment of inertia and undamped period of the coil, and from equation (26) the value of M, we find, upon inserting them in equation (41),[1] and applying equation (13),

$$K = \pi\rho^{1/\pi} \tan^{-1} \pi/\Lambda \sqrt{\frac{I_0 h(\lambda)}{10^7 r_1 T(\Lambda_1{}' - \lambda')}} . \qquad (45)$$

In this equation Λ represents the logarithmic decrement under the conditions under which the galvanometer is to be used. $\Lambda_1{}'$ and λ' have been explained in connection with equation (25). When the galvanometer is to be used on open circuit, $\Lambda = \lambda$; and when damping on open circuit is small, $\lambda' = \lambda$; the preceding equation may then be written

$$K = \pi \sqrt{\frac{I_0 \rho}{10^7 r_1 T(\Lambda_1{}' - \lambda)}} . \qquad (46)$$

In both equations (45) and (46) the value of the constant is given in coulombs per radian.

c) Third Method.--The value of M may be taken from equation (29) instead of from (26), as in the preceding derivation. The formulae analogous to (45) and (46) are then

$$K = \pi\rho^{\frac{1}{\pi}} \tan^{-1} \frac{\pi}{\Lambda} \sqrt{\frac{I_0 h(\lambda)}{10^7 r_c T(\pi - \lambda')}}, \qquad (47)$$

and

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[1] Presented before American Physical Society, Dec, 1913.

when Λ and λ are both small,

$$K = \sqrt{\frac{\pi\rho I_0}{10^7 r_c T}} \, . \tag{48}$$

2. Critically Damped Motion.--To find the general expression for the ballistic constant in this case, we proceed as before, for periodic motion. Substitution of equation (16) in (14) gives the value of the throw

$$\alpha_c = \frac{M}{\sqrt{I_0 q^2}} \frac{1}{\epsilon} Q, \tag{49}$$

from which
$$K_c = \frac{\sqrt{I_0 q^2}}{M} \epsilon . \tag{50}$$

A comparison of (50) with (41) indicates that in the critically damped case the formulae for the ballistic constant will be identical with those of the corresponding methods in the case of periodic motion, except that the damping factor in the latter is replaced by ϵ, its value for the boundary condition of damping. We may therefore write at once, corresponding to equations (43), (45) and (47), respectively,

$$K_c = t_c \epsilon k = \frac{T_0 \epsilon k}{2\pi} \, , \tag{51}$$

given in coulombs per centimeter when k is expressed in amperes per centimeter.

$$K_c = \pi\epsilon \sqrt{\frac{I_o h(\lambda)}{10^7 r_1 T(\Lambda_1' - \lambda')}} ; \tag{52}$$

$$K_c = \pi\epsilon \sqrt{\frac{I_o h(\lambda)}{10^7 r_c T(\pi - \lambda')}}, \tag{53}$$

or, if λ is small compared with π,

$$K_c = \epsilon \sqrt{\frac{I_o \pi}{10^7 r_c T}} . \tag{54}$$

3. Calculation of Ballistic Constant on Closed Circuit.--Equations (41) and (50) show that, for a given galvanometer, the ballistic constant varies only with the damping factor. When, therefore, the discharges to be measured take place in simple circuit, the galvanometer not being shunted, the equations developed in this section are directly applicable; the damping factor is determined from the logarithmic decrement corresponding to the resistance in the circuit.

From the above consideration a useful relation is obtained by taking the ratio between the constants corresponding, respectively, to two different conditions of damping. We then have

$$K_1 = K_o \frac{\frac{1}{\rho\pi} \tan^{-1} \frac{\pi}{\Lambda_1}}{\frac{1}{\rho\pi} \tan^{-1} \frac{\pi}{\lambda}}, \tag{55}$$

where K_1 is the constant corresponding to the logarithmic decrement Λ_1 which, in turn, corresponds to the resistance r_1 in the circuit, and where K_o is the constant on open circuit, with λ the corresponding logarithmic decrement. As before, if damping on open circuit is slight, the denominator of the fraction feduces to $\sqrt{\rho}$.

Likewise, when the resistance in the circuit is such that damping is critical,

$$K_c = K_o \frac{\epsilon}{\sqrt{\rho}}, \tag{56}$$

damping on open circuit being assumed small.

In these two equations, we may suppose K_o known, having determined its value, let us say, by means of a standard condenser charged by a standard cell. Further, let it be supposed that the value of the logarithmic decrement corresponding to a certain resistance is known. From these data and equation (27) we are able to determine all the factors in (55), consequently we may calculate the value of K_1 to be used with the apparatus of given resistance. Similarly, we may find K_c, in which case we proceed to find the critical damping resistance according to equation (30). The difficulty of making a direct determination of critical damping resistance has been mentioned.[1]

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[1]Page 19 of this paper.

B. Experimental.

Equations (48) and (54), on account of the difficulty of making a direct determination of the critical damping resistance, will not give reliable values for the ballistic constant; it was therefore considered superfluous to subject them to experimental verification. Equation (44) with a modification will be discussed at greater length in the next section of this paper. The experiments here to be described are upon formulae (46) and (55) in particular.

To apply equation (46), a galvanometer of the usual kind was used, with an upper suspension consisting of a 1.5-mil phosphor bronze strip. The period of the coil was 14.087 seconds, and its moment of inertia, found by comparison of its period, when suspended by a phosphor bronze wire, with that of an accurately turned brass disk of about the same mass, suspended by the same fiber, was 3.252 c.g.s. units. The logarithmic decrement on open circuit was 0.0384. Four different values of r_1 were taken, namely 5000, 8000, 10,000, and 15,000 ohms, to which corresponded, respectively, the four values of Λ_1': 0.3315, 0.2212, 0.1848, 0.1358. The resulting values of the ballistic constant in coulombs per radian were 1.270, 1.271, 1.271, 1.272, all multiplied by 10^{-5}.

The constant so determined was used in finding the capacity of a mica condenser, which was charged from a standard cell and discharged through the instrument. The value for the capacity was found to be 0.4960 microfarad; the capacity of the same condenser is given as 0.4656, obtained by Anderson's method of comparison with a self-inductance coil, the coefficient of which is known from a Bureau of Standards determination. Thus an agreement of 0.1 per cent. is indicated.

Another test of the same formula was made in the following manner: Each of two observers, K and Z, determined the constant of another galvanometer, the former by the method of equation (46), the latter by means of a standard condenser. K used the ordinary scale attached to the instrument, while Z used a straight scale and telescope at a greater distance from the instrument. The value found by K was 2.520×10^{-5}, that obtained by Z, 2.521×10^{-5} coulomb per radian. This close agreement must not, however, be taken to indicate the reliability of the method. From a large number of such comparative determinations which the author has made upon various instruments, the reliability may be stated as 0.15 percent.

Equation (55) was tested by finding K_0 by means of a standard condenser, and determining λ on open circuit in the usual manner, and then finding Λ_1 for the closed circuit of 4000 ohms resistance; this resistance was that of the galvanometer, of one of the coils of a standard mutual inductance, and of a resistance box, connected in series. The resulting value of K_1 found by the formula was 1.476×10^{-7} coulombs per cm. Throws obtained by breaking the primary circuit of the inductance coil, which was carrying an accurately measured current, gave a value for the constant of 1.473×10^{-7}.

A test of the method for finding the constant in the case of critical damping produced by the proper amount of resistance in the circuit with the coil amounts essentially to a test of equation (30). The results of such a test are given in Sec. VIII.

SEC. VI. ON THE CURRENT-DEFLECTION METHOD FOR DETERMINING BALLISTIC CONSTANTS OF MOVING COIL GALVANOMETERS, WITH A NOTE ON THE NON-UNIFORMITY OF MAGNETIC FIELDS IN SUCH INSTRUMENTS.

A. Theoretical.

The determination of the quantities involved in equations (43) and (44) is easily accomplished; and simple apparatus, found in any laboratory, is used. Let k in these equations be expressed in amperes per centimeter; in terms of the symbols, i/d.

a. Objection to the Use of K as Ordinarily Determined, and Proposal of Modified Method.--It seems that these equations are not commonly used, except for instructional purposes in connection with the theory of the ballistic galvanometer, the presumable reason being that the values which it gives for the constant are not often in agreement with those given by other methods, the difference amounting from a fraction of one percent. to as much as three or four percent., even when the work is carefully done. The major part of the error when damping is small occurs in the value of k; with strong damping, large errors are as likely to occur in Λ and in k. To this latter point reference will be made later. The formulae presuppose ideal conditions as to uniformity of field and symmetry of position of the coil with respect to the magnet; these conditions are far from being realized in most galvanometers, as has previously been indicated by the writer[1], and as will be shown at greater length in this paper. Since, ordinarily, the field is by no means uniform, the value of k is not truly constant, but depends upon the angular position of the coil.

The value of k to be used in the formulae is the value which depends upon the average field intensity <u>at the null position of the coil</u>, which, of course, is the position occupied by the coil at the instant the discharge passes through it. This fact suggests that it might lead to greater accuracy if, instead of producing a given deflection from the ordinary null position by means of a current, one were to produce a deflection by turning the torsion head from which the coil is suspended through a certain angle, and then to determine the current necessary to bring the coil back to its

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[1]Phys. Rev., N.S., 5, 266, 1915.

null position. The coil would then evidently be in the very position in the field which it occupies at the moment of the ballistic impulse. The question at once arises whether the current constant should, with this manipulation, have the same value as that which would be obtained by the ordinary method if the field were truly uniform; for it is evident that in this procedure the spiral lower suspension would, when the coil were returned to its null position, be in the same condition of strain as it was before the coil was deflected by turning the torsion head.

To investigate this point, we may proceed as follows: Let i_1 and i_2, respectively, be the currents through the coil equivalent to the torque per unit angle of the upper and lower suspensions; and let the upper torsion head be turned through an angle d_o (in terms of centimeters on the circular scale), producing an angular deflection d of the coil. Then

$$i_2 d = i_1(d_o - d),$$

from which

$$d = i_1 d_o/(i_1+i_2).$$

Now let a current i be sent through the coil, just sufficient to bring it back to the original null position; since the lower suspension is now free from strain,

$i = i_1 d_1$; consequently

$$k = i_1 + i_2.$$

The same result follows when the assumption is made that the lower suspension is given sufficient twist to produce the deflection d of the coil. By twisting either suspension, therefore, the restoring couple of the system remains unchanged. This leads to the important conclusion in connection with the usual condition[1] of the suspensions in galvanometers, that within limits, no matter what the condition of strain of the suspensions at the null position, the elastic constant of the suspensions is the same as though the suspensions were free from strain.[2]

b. Precautions to be Observed in Determining k by the Proposed Method.--In determining the current constant by the modified method, one source of error should be kept in mind, namely, that in most instruments--unless special attention is paid to this point--the prolonged axis of the coil does not pass through the point of attachment of the upper suspension. This may,

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[1]This condition being that, at the null position of the coil, one of the suspensions is twisted, necessitating a twist of the other suspension in the opposite direction for equilibrium.

[2]Pealing (Phil.Mag. 29, 203, 1915) shows that "bifilar effect" of a phosphor bronze ribbon is negligible in its effect upon the elastic torque constant.

for example, be due to a bent terminal, or to eccentricity of the torsion head. When the torsion head has been turned through a certain angle and the coil brought back to zero deflection by means of a suitable current, the position of the coil is usually not identical with the original null position, but may be shifted into a region of the field at which the intensity differs appreciably from that at the original null. The current constant will then be affected by an error proportional to the difference between the field intensities at the two positions. The error from this cause, for a given angular displacement of the coil, is less with an upper suspension having a large torsional moment than with a fiber of which the torque is small; for in the latter case the upper torsion head must be turned through a larger angle to produce a given deflection of the coil than in the former. The gain in accuracy is, however, offset by the smaller sensitivity with a stiff upper suspension. It should therefore be observed, particularly when a galvanometer with weak control is being used,that the upper suspension be as nearly coicident with the vertical axis of symmetry of the instrument , and the coil as near the ideal position, as careful levelling can bring them.

Another precaution to be observed is that of using the proper value of the logarithmic decrement correcponding to a given throw, especially when there is considerable damping; since the logarithmic decrement is not constant,[1] its value must be determined for any given amplitude. A method for doing this is described in the experimental part of this section.

B. Experimental.

1. Tests on uniformity of magnetic field.

a. By Means of Current Sensitivity.--An experiment was first carried out for the purpose of gaining some idea as to how inhomogeneity of the magnetic field may affect current sensitivity; more particularly, how it may affect the accuracy of k as found by the proposed method when there is eccentricity in the upper torsion head.

A 3-mil phosphor bronze strip upper suspension was selected with terminal wires such as are provided by the manufacturers, about 8 mm. long. The upper terminal wire was bent slightly in order to produce the eccentricity before mentioned. The shift is represented by the radius indicated in the dotted circle A' of Fig. 1,

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[1]Peirce, B. O., Contributions from the Jefferson Physical Laboratory, 6, 64, 1908.

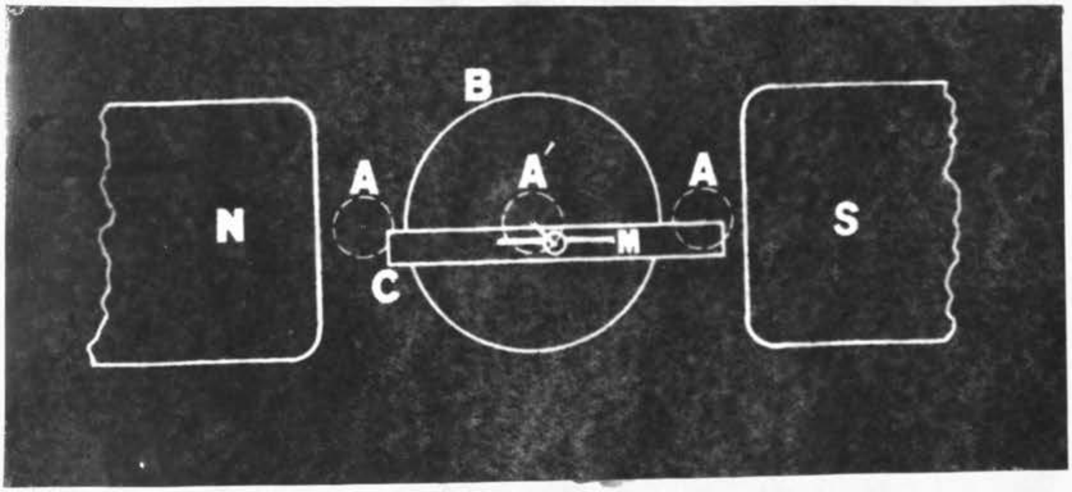
B
A
A'
A
N
S
M
C

Fig. 1.

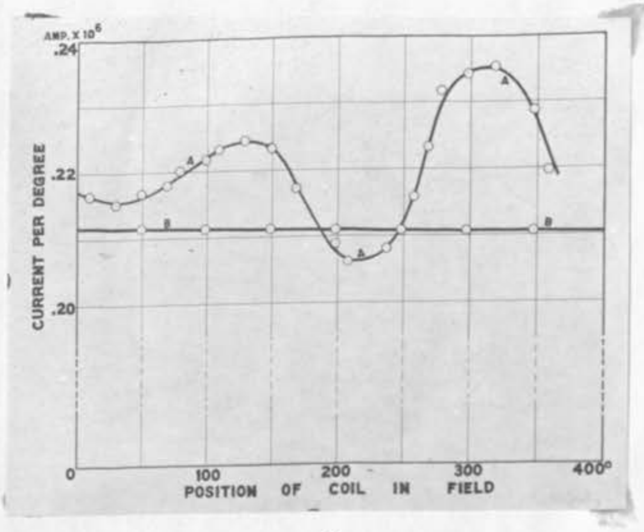
AMP. X 10^6
.24
.22
.20
CURRENT PER DEGREE
A
B
0
100
200
300
400°
POSITION OF COIL IN FIELD

Fig. 2.

the actual radius being about 2mm. The figure gives a view from above of the coil in the magnetic field. N and S are the poles, B the soft iron core, and C the coil which carries the mirror M. If the plane of the coil is kept parallel with the original position while the upper torsion head is turned through 360 degrees, the horizontal projection of the axis of the coil describes the circle A' while the mid-points of the sides describe the circles A, A, each equal to A'. The experiment was carried out by turning the upper torsion head through a small angle and sending sufficient current through the coil to bring it back to the null reading. The angle was read on a large divided circle attached to the torsion head, and the current was determined by means of a potentiometer. The process was then repeated a sufficient number of times to make a complete excursion about the circles A, A', A. Curve A, Fig. 2, is the graphical result of thes observations. The abscissas of this curve are in terms of degrees which the radius drawn in A' made with an arbitrary initial position. The higher second maximum of the coil indicates that at this position the coil was forced into a still weaker field than at the first maximum, possibly by a slight buckling of the upper suspension. By taking different initial positions of the coil, one can at will shift the maxima horizontally; and by tilting the instrument forward or back slightly, the sensitivity for any one position of the coil is changed.

Curve B, Fig. 2, show the results obtained when the suspension terminals were very short, so as to admit of no such eccentricity as in the case of curve A. Again, by tilting the instrument forward or back, it ~~was~~ is possible to vary the constant within fairly wide limits; but no marked change of sensitivity results merely from twisting the upper suspension. Within the limits of the scale at 50 cm. the change is imperceptible. This goes to show that the elastic constant of the suspensions is, within the limits of observational accuracy and at the elongations ordinarily used, unchanged by twisting.

Similar results were obtained with a 1.5-mil strip suspension, with the difference that the maxima and minima in the sensitivity curve showed no such variation as in-dicated in curve A for the heavier suspension.

b. By Means of Ballistic Sensitivity.-- The results of the preceding experiment make it seem probable that the differences which one frequently observes in the ballistic throws[1] of an instrument with equal quantities of electricity but in opposite directions from the same null point may be ascribed to the non-uniformity of the galvanometer field. To investigate this point an experiment suggests itself, as follows: Using an upper suspension fiber with short terminals, so as to avoid eccentricity, one may, by means of the upper torsion head, shift the coil into various angular positions with respect to its usual null position; for each particular position one may take throws in opposite directions with equal quantities.

This was done, starting with a shift of about 10 degrees towards the "red" side of the scale, and, in small steps, working over to a position of about the same displacement on the "black" side. The results are shown graphically in Fig. 3. The instrument was moderately damped, and the same quantity of electricity was used for every throw. Abscissas are given in degrees of displacement of the various positions from the ordinary null, ordinates represent centimeters of throw from each position. "B" refers to the "black" side of

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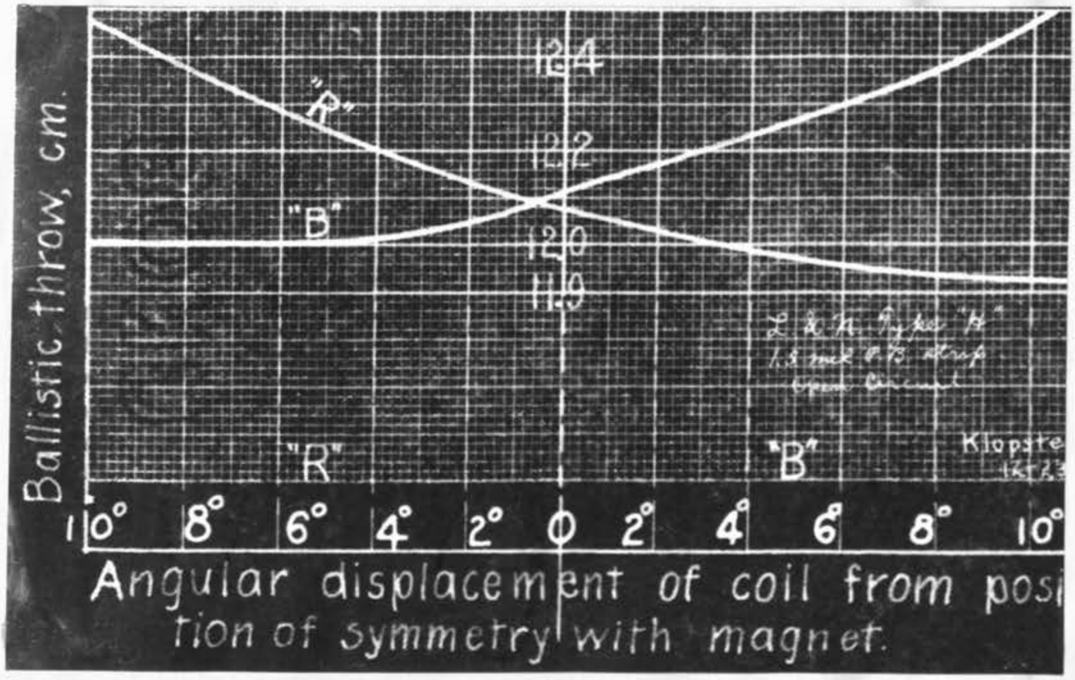
[1]Peirce, B. O., Am. Acad. Proc., 42, 161, 1906.

the scale, "R" to the "red" side. The two curves in the figure are seen to cross at a point about 0.8 degree from the null towards the "red", indicating that if one were to shift the coil into this position as a new null, equal throws would be obtained with this particular quantity. With larger or smaller quantities the curves would not necessarily intersect at the same abscissa.

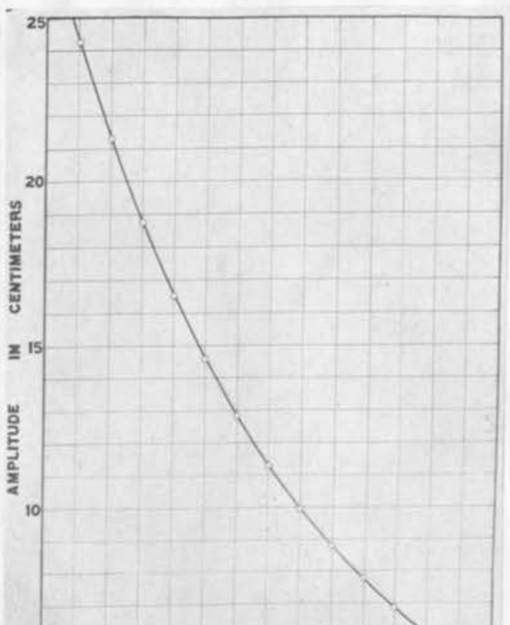
Clearly, with a given quantity, the magnitude of the throw depends upon two factors: the intensity of the field at the null position of the coil, and the amount of damping in the region through which the coil swings. The former is the same for any given position of the coil, regardless of the direction of swing; the latter depends upon the nature of the magnetic field. The experiment shows non-uniformity in two ways: first, differences in throws in the same direction from different null positions; and second, differences between throws in opposite directions from the same null.

The foregoing experiments indicate that for every possible position of the coil in the magnetic field both deflection and ballistic constants may have different values. This suggests that a galvanometer which is moved about at all, if its sensitive-



Ballistic throw, cm.
12.4
12.2
12.0
11.9
"R"
"B"
"R"
"B"
Klopste
10° 8° 6° 4° 2° 0 2° 4° 6° 8° 10°
Angular displacement of coil from posi
tion of symmetry with magnet.

Fig. 3.



AMPLITUDE IN CENTIMETERS
25
20
15
10
5
0
5
10
15
TIME IN TERMS OF COMPLETE PERIOD OF COIL

Fig. 4.

ness for any purpose with a particular fiber is to remain unaltered, or very closely so, should be provided with some sort of an adjustable level indicating device, such as a small plumbline with a cylindrical bob hanging through a hole in a horizontal plate, the diameter of the hole being, perhaps, a half millimeter greater than that of the bob. When the instrument has been levelled as desired, the point of suspension of the plumbline may be so shifted that the hole and cylindrical plumb-bob are concentric, and clamped in this position. This constitutes a permanent indicator, according to which the instrument may be re-levelled after it has been disturbed.

2. Method of finding logarithmic decrement to be applied to throw of given magnitude.

When damping is slight, it is quite sufficient to find the logarithmic decrement or damping factor in the usual manner, using the values of successive elongations of the order of magnitude of the throw to be corrected for damping; remembering that these elongations are to be taken on the same side of the null as the throw, on account of probable differences in damping on the two sides. For highly damped throws,

--the motion of the coil still being periodic--the above method may not give the value with sufficient accuracy because of the great difference between two successive elongations on the same side of the null. The procedure in this case is illustrated by means of the curve of Fig. 4.

The coil is started swinging, the resistance in the circuit being the same as that which is to be used in the measurements to follow. As many sets of readings of successive elongations, on the side of the scale to be used, are taken as is considered necessary for the desired degree of accuracy. One of these sets is first plotted, as in Fig. 4, using as abscissas the time, expressed in terms of the complete period of the coil as a unit; the figure is simply the locus of the maximum values of the swings on one side of the scale. The other sets may then be "fitted in", using the same unit, and the curve drawn. To apply the curve to the determination of ~~any~~ the logarithmic decrement corresponding to any particular amplitude, we find the intersection of the curve with the horizontal line at the amplitude in question; from this point we pass to the left a distance equal to half of one scale unit, and find the amplitude corresponding to this time. From these two values the logarithmic decrement is determined. ~~question~~. This damping correction will be slightly affected by the zero shift due to magnetic impurities[1]; for in an actual measurement of quantity one eliminates the effect of zero shift as suggested by Zeleny, while the curve here described is obtained from readings of the coil as it executes complete vibrations. The amount of the error has not been determined, but it would seem that in the average instrument it must certainly amount to less than 0.1 percent.

3. Test of modified method for obtaining K.

The results shown in Tables 2 and 3 were obtained in a series of determinations, the purpose of which was not so much to approach the limit of accuracy attainable by this method of finding the ballistic constant as to determine the reliability of the method under ordinary laboratory conditions. Each quantity in the formula was determined from but a single careful observation. The instruments were carefully levelled, but it was not attempted to eliminate absolutely the possible eccentricity due to bent terminals. The galvanometers are the ones which are constantly used in the undergraduate

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[1]Zeleny, A., Phys.Rev., O.S., 23, 400, 1906; ibid., 32, 297, 1911.

laboratory; no tests were made to see if the scales were exactly circular, or if they were accurately 50 cm. from the mirrors.

A comparison of the results shows that even without these precautions the modified method is superior to that commonly used. In the writer's opinion one is justified, when careful work is done and several determinations of each quantity are made, in assuming an accuracy of 0.2 percent. in the results.

It would greatly facilitate the manipulation of returning the coil to its normal position from the deflection produced by turning the upper torsion head if the latter were provided with a tangent screw, permitting accurate adjustment at the first trial.

SECTION VII. THE CORRECTION FOR THERMOELECTRIC CURRENT TO BE APPLIED TO THE THROWS OF A BALLISTIC MOVING COIL GALVANOMETER.

A. Theoretical.

¶ On account of the simplicity and convenience of the condenser method of determining the ballistic constant of a moving coil galvanometer, one is likely to use it in preference to the more complicated inductance coil methods in making an accurate determination of the constant. This is especially true when the instrument is used on open circuit, in which case there are no particular difficulties to be overcome in making a precise measurement. When the discharge to be measured takes place in the closed galvanometer circuit,[1] the constant should be determined under the same or equivalent conditions, because of the dependence of ballistic throw upon damping. When using a standard condenser for obtaining the constant on closed circuit it is necessary, immediately after the condenser has been

[1]Zeleny and Erikson, Manual of Physical Measurements, 3d ed., 172; Smith, Electrical Measurements, p. 194.

discharged, to close the circuit through a resistance equal to the resistance of the apparatus to which the galvanometer is to be connected in making the measurements. This is easily accomplished by means of special keys or switches.

Upon changing from open to closed circuit one is likely to encounter a thermoelectric current[1] which deflects the coil either in the same direction as or opposite to that of the throw caused by the impulsive discharge. The time of reaching the elongation from the null point is different, however, in the case of a deflection due to a steadily impressed electromotive force from that in the case of a current of very short duration. Were these time intervals alike, the correction could be made by algebraic addition of the maxima. The relation between them depends upon the amount of damping in the circuit, and is therefore different for different values of the logarithmic decrement. Zeleny[2] has given an experimental curve for correcting the observed throw and obtaining from it the magnitude of the throw which would have resulted in the absence of the thermoelectromotive force.

It is the purpose of this paper to obtain a relation between the quantities involved, by means of which the correction may be made.

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[1]Zeleny, A., Phys.Rev.,O.S., 23, 414, 1906.--The method of equation (55) obviates the necessity for making the correction for thermoelectric effect.

[2]Loc. cit.

If we set θ' in equation (22) equal to zero and solve the equation for t, we find the value of the interval from the instant the coil was set in motion by a steady current to the instant of reaching its maximum displacement. This is found to be

$$t = \pi/b = T/2. \tag{57}$$

This value of t substituted in (21) gives the value of the maximum displacement, α_d:

$$\alpha_d = \frac{Mi}{q^2}(1 + \epsilon^{-\Lambda}). \tag{58}$$

The time of elongation, when the coil is set in motion by an instantaneous discharge is given by equation (6). Both equations (6) and (57) may apply to the same galvanometer coil, provided the values of a and b are the same in each equation.

Suppose, now, that a quqntity of electricity Q in a condenser is discharged through the galvanometer and that the circuit is immediately closed; and that, upon closing, a thermoelectric current flows. If the direction of the steady current is the same as that of the discharge, the resulting maximum is greater than it would have been in the absence of the thermoelectromotive force; if the direction of the current is opposite that of the discharge, the resulting throw is smaller. Obviously, the method for finding the correction to be applied to the observed throw is, by the use of equations (6) and (21), to determine the angle attained in a steady deflection during the time interval required for a ballistic maximum.[1] Designating this angle by ω, we find

$$\omega = \frac{Mi}{q^2}\left[1 - \epsilon^{-\frac{a}{b}\tan^{-1}\frac{b}{a}}\left(\cos\tan^{-1}\frac{b}{a} + \frac{a}{b}\sin\tan^{-1}\frac{b}{a}\right)\right], \qquad (59)$$

or, in terms of the logarithmic decrement,

$$\omega = \frac{Mi}{q^2}\left(1 - \rho^{-\frac{1}{\pi}\tan^{-1}\frac{\pi}{\Lambda}} \cdot \frac{2\frac{\Lambda}{\pi}}{\sqrt{1 + \frac{\Lambda^2}{\pi^2}}}\right). \qquad (60)$$

In order to make this equation for the correction applicable to any galvanometer of the type under discussion, it is convenient to express ω in terms of either of two ratios: $R_1 = \omega/d$, giving its value as a fraction of the steady deflection produced by the thermal current; or, $R_2 = \omega/\alpha_d$, which gives the value as a fraction of the first maximum due to the steady current. The value of d from equation (21) is seen to be Mi/q^2; consequantly

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[1]Assuming the steady deflection small in comparison with the throw. Zeleny mentions 1/3 as the maximum permissible ratio. For further discussion see below.

$$R_1 = 1 - \rho^{-\frac{1}{\pi}\tan^{-1}\frac{\pi}{\Lambda}} \frac{2\frac{\Lambda}{\pi}}{\sqrt{1 + \frac{\Lambda^2}{\pi^2}}} . \tag{61}$$

Comparing equations (60) and (58), and noting that $\log_\varepsilon \rho = \Lambda$, it is seen that

$$R_2 = R_1 \frac{\rho}{1 + \rho} . \tag{62}$$

Inasmuch as the actual motion of the coil is the result of superposition of the motions defined by (4) and (21), the exact equations of the two possible kinds of motion are obtained, respectively, first, by adding the two equations, and second, by subtracting the one from the other. The observed maximum then occurs not at the instant given by equation (6), but a trifle sonner or later than this, depending upon the relative directions of the current and impulsive discharge. By equating to zero the angular velocities obtained by differentiating each of the combined equations and solving for t, we may write the result for both cases:

$$t = \frac{1}{b} \tan^{-1} \frac{b}{a \pm \frac{i}{bQ}} \tag{63}$$

The upper or lower sign is taken according as the current and discharge are in the same or in opposite directions, respectively.

It would be a rare occurrence in an ordinary measurement to encounter an extraneous thermoelectric current which is appreciable compared with bQ,[1] considered in its effect upon the time required for the throw. Should it occur, however, the procedure would be to find the correction by several approximations, using equation (63) instead of (6) in equation (21). To substitute (63) directly in the equations of motion obtained by the addition or subtraction of (4) and (21), and then to form the ratios to be used for correcting the observed throws would lead to expressions which are too complex for practical application.

B. Experimental.

a. Procedure in Correcting Observed Throws.--Consideration of equations (61) and (62) shows that for a determination of either R_1 or R_2 for any particular case a single experimental determination suffices, namely of the quantity ρ. The factors $\rho^{\frac{1}{\pi}\tan^{-1}\frac{\pi}{\Lambda}}$ and $h(\Lambda)$ which occur in the equations are obtainable from tables[2], which makes the formulas readily applicable,

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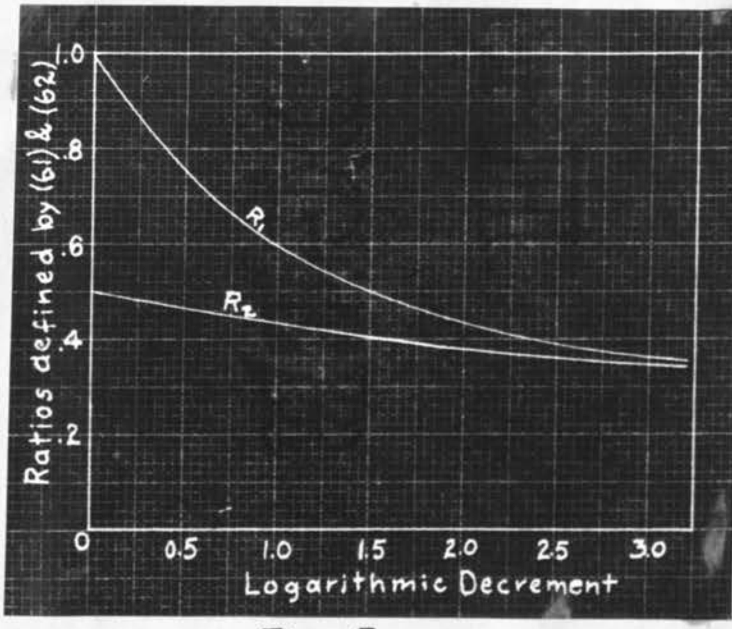
[1]The quantity $1/b$ has the dimension time, so that bQ has the dimensions of a current.

[2]Kohlrausch, 12th ed., p. 723.

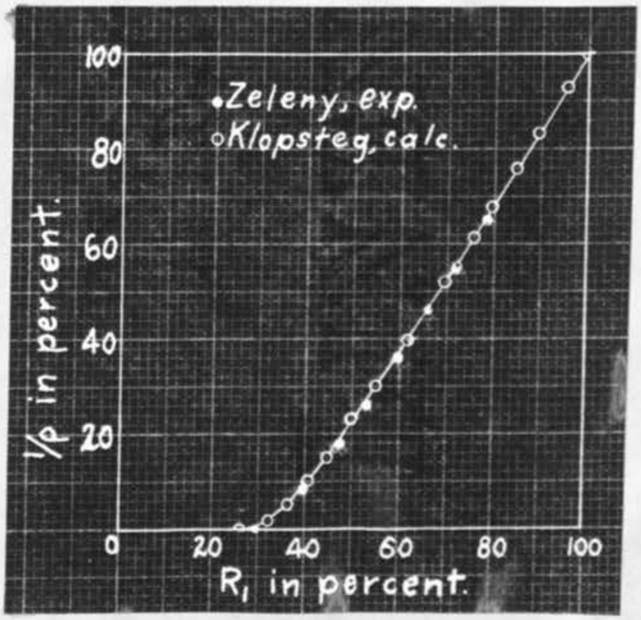
To use the correction given by equation (61), the observed steady deflection due to the thermoelectric current which flows when the circuit is closed is multiplied by R_1 and the result is added to or subtracted from the observed throw, depending upon the relative directions of current and discharge; to apply equation (62), the first maximum amplitude due to the thermoelectric current only is observed, and the correction obtained by multiplying this value by R_2. Having once obtained R_2 for a particular condition of damping, its application is more economical of time than the use of R_1, since the first maximum elongation may be observed T/2 seconds after closing the circuit, obviating the necessity for waiting until the deflection has attained its steady value, or of using a short-circuiting key for bringing the coil to rest.

The curves of Fig. 5 show the variation of each of the ratios with logarithmic decrement. In these curves the abscissas have been carried to $\Lambda = 3.2$ only, inasmuch as, in practice, Λ seldom exceeds 1 when the galvanometer is used ballistically; if the logarithmic decrement is large, it is most convenient to render the coil just aperiodic by diminishing the resistance in the circuit



Ratios defined by (61) & (62)
1.0
.8
.6
.4
.2
0
0.5
1.0
1.5
2.0
2.5
3.0
Logarithmic Decrement
R_1
R_2

Fig. 5.



•Zeleny, exp.
∘Klopsteg, calc.
1/p in percent.
100
80
60
40
20
0
20
40
60
80
100
R_1 in percent.

Fig. 6.

according to equation(30). In the critically damped case $R_1 = R_2 = 26.4$ percent. This is shown in Fig. 6.

b. Test of the Equations.--The experimental test of the method here outlined was taken from the work of Zeleny.[1] From equation (61) a set of corresponding values was obtained for R_1 and $1/\rho$, the abscissas and ordinates, respectively, of the experimental curve to which reference has been made. Fig. 6 shows the curve obtained by plotting these values, together with representative points taken from Zeleny's curve. There is good agreement between the theoretical and experimental curves.

[1] Loc. cit.

SECTION VIII. APPLICATION OF THE MOVING COIL GALVANOMETER TO THE MEASUREMENT OF TIME.

I. Introduction.

The well known method of measuring the time occupied by events of very short duration by means of the ballistic galvanometer[1] depends upon the relation between electrical quantity and current, and upon the fact that for sudden discharges the throws of the galvanometer coil are proprotional to the quantities which have passed. Essentially this is the method described by Brown[2], who used an unbalanced Wheatstone net[3] for applying the potential difference to the galvanometer terminals. The observed proportionality between throws and time intervals up to 3 seconds was obviously due to the fact that his galvanometer had a period of 91 seconds. The reliability of his absolute measurements was about 2.5 percent., the amount of the discrepancy between the two values of the ballistic constant as determined by two different methods.

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[1] Kohlrausch, Lehrb. d. prakt. Phys., 12th ed., 527.

[2] Brown, F. C., Phys. Rev., O.S., 34, 452, 1912.

[3] The Wheatstone net is not essential to the method. A simpler means of applying the potential difference to the galvanometer would simplify his equations (5) and (6), p. 453.

When an event lasting but six or seven seconds is to be measured with a fair degree of accuracy, a stop watch, because of its mechanical limitations, and the large probable error in starting and stopping, is out of the question. It is not to be gainsaid that a simple method of measuring intervals of several seconds' duration--one which might, for example, be used with confidence in place of a chronoscope or a high speed chronograph with a standard clock--would prove of value, especially in laboratories where the standard apparatus for the measurement of short intervals is not a part of the equipment. Brown's suggestion of reserving a slow period galvanometer for the purpose primarily meets with the objection that such an instrument, because of its sluggish motion, and because of the attention required to bring the coil to rest, is almost certain to lose one much time. In this regard a short period galvanometer--one, say, of 10 to 20 seconds' period, such as is commonly used in our laboratories,-- would be much better; in fact, inasmuch as high sensitivity is not a requirement, the best sort of instrument to use, on account of its "self-adjustment" to the zero position, would seem to be the short period instrument, damped critically by means of a shunt of proper resistance.

To employ a short period instrument for measuring time intervals, then, which are greater than those during which the coil has not moved appreciably[1], we must impress upon the instrument a steady current during the interval to be measured, which clearly means that the current shall be flowing while the coil is moving. The problem is, from an elongation so obtained, to deduce the length of the interval during which the current was caused to flow. Dorn[2], Diesselhorst[3], Peirce[4] and Worthing[5] have treated the inverse of this problem for quantities obeying various laws of discharge, the two last-named for the moving coil type of instrument.

II. Theoretical.

A. General Outline of Method of Derivation.

In the development of the theory, the assumptions as to type of instrument which were enumerated at the beginning of Sec. I. will be adhered to. The method of deduction, for any condition of damping, will be as follows:

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[1]This is the assumption upon which the ordinary ballistic measurement, in which throws and quantities are taken proportional, is based.

[2]Dorn, E., Wied.Ann., 17, 654, 1882.

[3]Diesselhorst, H., Ann.d.Phys., 9, 712, 1902.

[4]Peirce, B. O., Am.Acad.Proc., 44, 283, 1909.

[5]Worthing, A. G., Phys. Rev., N. S., 6, 165, 1915.

1. From the equation of its motion, the position and angular velocity of the coil, initially at rest, are determined for a particular instant, τ seconds after a steady current has begun flowing through the instrument.

2. These respective values of position and angular velocity are introduced as initial conditions into the equation of motion of the coil, swinging without resultant torque, corresponding to the condition that at the instant τ the current was cut off.

7 3. From the resulting equation is found, in the usual manner, the maximum angle of displacement; and the expression for this angle is to be solved for τ.

On account of their presumable applicability the two cases of periodic and of critically damped motions, respectively, will be treated.

B. Damped Periodic Motion.

a. Derivation.-- Following the plan of procedure as outlined we have, using the value $t = \tau$ in equations (21) and (22), respectively,

$$\theta_\tau = \frac{Mi}{q^2}\left[1 - \epsilon^{-a\tau}(\cos b\tau + \frac{a}{b}\sin b\tau)\right], \qquad (64)$$

and

$$\omega_\tau = \frac{Mi}{q^2}\,\epsilon^{-a\tau}\left[(b - \frac{a^2}{b})\sin b\tau - 2a\cos b\tau\right]. \qquad (65)$$

These equations represent the value of displacement and angular velocity at the instant the current is shut off. We must now determine the elongation which the coil will attain by virtue of its angular momentum.

Equation (3), when β is imaginary, may be written in the form

$$\theta = \epsilon^{-at}(A \cos bt + B \sin bt), \tag{66}$$

where a and b have the same values as before, because experimentally it is possible so to arrange the circuits that exactly the same conditions of damping obtain when the electromotive force is applied as when it is shut off. A and B are constants of integration, which are to be evaluated by introducing the conditions that when $t = 0$, $\theta = \theta_T$ and $d\theta/dt = \omega_T$. The equation then becomes

$$\theta = \epsilon^{-at}\left(\theta_T \cos bt + \frac{\omega_T + a\theta_T}{b} \sin bt\right). \tag{68}$$

By the usual method it is found that the direction of motion of the coil reverses for the first time at the instant

$$t_1 = \frac{1}{b} \tan^{-1}\left[\frac{\omega_T b}{a\omega_T + (b^2+a)\theta_T}\right], \tag{69}$$

and that the corresponding elongation (designating the factor within the brackets by ψ) is

$$\beta_T = \epsilon^{-\frac{a}{b}\tan^{-1}\psi}\left(\theta_T \cos \tan^{-1}\psi + \frac{\omega_T + a\theta_T}{b} \sin \tan^{-1}\psi\right). \tag{70}$$

In the preceding equation β_τ denotes the angle of "throw" due to a steady current flowing for τ seconds; β_i will signify the throw due to the same quantity instantaneously discharged, both applying to damped periodic motion. Similarly, α_τ and α_i will represent the same quantities for undamped motion, and γ_τ and γ_i for the case of critical damping. Obviously, we might now proceed to substitute, first for ψ, and then for θ_τ and ω_τ, the values given by equations (69), (64) and (65), respectively; the resulting equation would involve τ, the quantity sought, θ_τ, the quantity observed, together with other determinable factors. However, the equation would not express the value of τ explicitly, which would necessitate a graphical method or one of approximations to find the interval. Hence, to obtain a practicable formula explicitly solved for τ, we may, for purposes of derivation, put $f = 0$ in the preceding equations, and reintroduce the correction for damping after the final equation is found. If $f = 0$, $a = 0$, and $b = \sqrt{q^2/I_0}$. With this simplification equation (70) reduces to

$$\alpha_\tau = \frac{Mi}{q^2}\, 2 \sin \frac{b\tau}{2}, \tag{71}$$

from which

$$\tau = \frac{T_0}{\pi} \sin^{-1} \frac{k}{2i}\, \alpha_\tau. \tag{72}$$

Introducing the damping correction, and assuming damping to be small,

$$\tau = \frac{T}{\pi} \sin^{-1}\frac{k}{2i}\sqrt[4]{\rho}\ \beta_{\tau}. \tag{73}$$

b. Discussion. Sources of Error and Precautions.--Formula (73) may be used as it stands for the measurement of time intervals--according to the equation--equal to a half period of the coil. The factor which gives rise to the greatest uncertainty in the result is the current constant[1], which is not the same at all deflections, principally on account of the distortion of the magnetic field and probable imperfect levelling of the instrument. However, the field about the coil in its usual equilibrium position is fairly uniform within a region corresponding to a deflection of a few centimeters on either side of the null position. The error in the constant may therefore be minimized by shifting the coil, by means of the upper torsion head, to a position about 6 cm. to one side of the usual null point, using this as a new zero setting; the deflections from this zero are then kept within 6 cm. beyond the original null, i.e., within 12 cm. This manipulation also largely eliminates the zero shift due to magnetic impurities in the coil.[2]

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[1] Page 42 of this paper.
[2] Zeleny, A., Phys. Rev., O.S., 32, 297, 1911.

c. Modified Form of Equation (73).--To obviate much of the labor of computation involved when equation (73) is used for determining a number of intervals of different durations, we may make this equation the basis of a graphical method. By equation (71) it is seen that, for zero damping, $\alpha_T/\phi = 2 \sin x$, where ϕ is the steady deflection produced by the same current which is used in obtaining the throw[1], and x is written in place of $\pi\tau/T_0$. With the aid of equation (44), equation (73) may be written

$$\alpha_i = \alpha_T \frac{x}{\sin x}, \tag{74}$$

which remains true when α is replaced by β. Thus, if we plot 2 sin x as abscissas and x/sin x as ordinates, we obtain a correction curve which enables us, from a given throw in terms of the angle of steady deflection with the same current, to obtain the correction factor by which this throw must be multiplied in order to give the throw which would have been produced by the same quantity instantaneously discharged. Inasmuch as the damping factor has not been introduced into the curve, the latter is applicable to any instrument--the observed throws being corrected for damping before the correction is applied.

[1]When short intervals are measured it is not feasible to produce this steady deflection directly on account of the large current which must be used to obtain a throw of sufficient magnitude. In this case ϕ is calculated from the current constant obtained with a small current.

Having found the "instantaneous throw", the damping correction is made, and τ found from the simple relation

$$\tau = \frac{T_0 \beta_i \sqrt{\rho}}{2\pi\phi}, \tag{75}$$

which may be put in the form

$$\tau = \frac{K}{I} \frac{x}{\sin x} \beta_i. \tag{76}$$

The latter form, in which K may be determined by the standard condenser method, does away with direct time determination.

d. Effect of Prolonged Discharge upon Throw.--Equation (74) indicates the error made in assuming a quantity of electricity, flowing uniformly, to have passed through the coil in a negligibly short interval, the discharge having actually been prolonged through an interval τ. The correction factor[1] is $x/\sin x$, which is independent of the damping factor over a wide range of conditions. It does not, however, apply with exactness to critically damped motion, as will be shown later.

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[1]The expression given by Diesselhorst (l.c.) for a prolonged steady current, reduced to the notation of this paper, is $\alpha_\tau = \alpha_i(1 - x^2/6)$. The corresponding form of equation (74) is $\alpha_\tau = \alpha_i(\sin x/x)$; expanding $\sin x/x$, we obtain $1 - x^2/3! + x^3/4! - \ldots$, which shows Diesselhorst's correction, obtained by a different method, to be the first term of the series given. This is sufficiently accurate for the shorter intervals.

Equation (74) shows the theoretical error committed in any measurement, when the measured interval is less than about T/30, to be less than 0.2 percent. Applying this to a galvanometer of 91 seconds' period, like the one used by Brown, the discharge may be allowed to pass a trifle over 3 seconds without introducing appreciable error into the measurement. This agrees with the results given in the paper cited.

C. Critically Damped Motion.

a. Derivation.--The same procedure will be followed as in the case of periodic motion, except that no assumptions as to damping need be made. The final formula is theoretically accurate.

Corresponding to equation (64), we have

$$\theta_T = \frac{Mi}{q^2}\left[1 - \epsilon^{-a\tau}(1 + a\tau)\right], \tag{77}$$

and to equation (65),

$$\omega_T = \frac{Mi}{q^2}\, a^2\tau\epsilon^{-a\tau}. \tag{78}$$

When the resultant displacing torque is zero, the equation of motion is

$$\theta = \epsilon^{-at}(A + Bt); \tag{79}$$

and, imposing the same initial conditions as upon equation

(66), to evaluate the arbitrary constants,

$$\theta = \epsilon^{-at}\left[\theta_T + (\omega_T + a\theta_T)t\right]. \tag{80}$$

The maximum of this function occurs at the instant

$$t_c = \frac{1}{a}\left(1 + \frac{a\theta_T}{\omega_T}\right) \tag{81}$$

and its value is, corresponding to equation (70),

$$\gamma_T = \epsilon^{-\frac{\omega_T}{\omega_T + a\theta_T}}(\theta_T + \omega_T/a). \tag{82}$$

Inserting the values of θ_T and ω_T, and expressing the result in terms of the ratio of the observed throw to the steady deflection with the same current,

$$\frac{\gamma_T}{\phi} = \frac{\epsilon^{2x} - 1}{2x\,\epsilon^{\left(\frac{\epsilon^{2x}}{\epsilon^{2x}-1}\right)}}, \tag{83}$$

where x has the same significance as before. For critical damping we have the relation

$$\gamma_i = \frac{Mi\tau}{q^2\epsilon t_1}. \tag{84}$$

Now let the factor by which the observed throw must be multiplied in order to reduce it to the equivalent throw resulting from an instantaneous discharge of the same quantity be represented by δ; then

$$\delta = \frac{2x}{\epsilon} \frac{\epsilon^{2x\left(\frac{\epsilon^{2x}}{\epsilon^{2x}-1}\right)}}{\epsilon^{2x}-1} \qquad (85)$$

From (83) and (85) are obtainable corresponding values of γ_T/ϕ and δ, for various assigned values of 2x. The curve of Fig. 7 is plotted with γ_T/ϕ as abscissas and δ as ordinates, and Table 4 gives the values from which such a correction curve may accurately be plotted. This method of treatment is unavoidable because of the implicit appearance of τ in equation (20).

b. Application of Method and Precautions.--The procedure in making a determination of an interval by this method is like that outlined for equation (74), page 71, except that the damping factor is now included.

Having determined γ_T/ϕ, δ is found from the table or curve; the product of these two quantities is γ_i/ϕ, which is substituted in

$$\tau = \frac{\epsilon \cdot T_0 \gamma_i}{2\pi \cdot \phi} . \qquad (86)$$

This equation may be written in a form analogous to (76):

$$\tau = K\gamma_i/i, \qquad (87)$$

which does away with the necessity of a direct determination of time. The precautions mentioned in connection

(To follow page 75)

2x	γ_T/ϕ	δ	2x	γ_T/ϕ	δ
0.00	0.0000	1.0000	1.60	0.5325	1.1055
0.05	0.0184	1.0002	1.65	0.5458	1.1120
0.10	0.0368	1.0004	1.70	0.5590	1.1189
0.15	0.0551	1.0010	1.75	0.5718	1.1260
0.20	0.0734	1.0017	1.80	0.5845	1.1330
0.25	0.0917	1.0026	1.85	0.5968	1.1404
0.30	0.1099	1.0038	1.90	0.6089	1.1479
0.35	0.1281	1.0051	1.95	0.6207	1.1557
0.40	0.1462	1.0067	2.00	0.6323	1.1636
0.45	0.1642	1.0084	2.05	0.6436	1.1717
0.50	0.1820	1.0104	2.10	0.6547	1.1801
0.55	0.1998	1.0126	2.15	0.6654	1.1887
0.60	0.2175	1.0149	2.20	0.6760	1.1973
0.65	0.2350	1.0175	2.25	0.6863	1.2061
0.70	0.2524	1.0204	2.30	0.6963	1.2152
0.75	0.2696	1.0234	2.35	0.7060	1.2246
0.80	0.2867	1.0266	2.40	0.7156	1.2337
0.85	0.3036	1.0300	2.45	0.7249	1.2434
0.90	0.3203	1.0336	2.50	0.7339	1.2532
0.95	0.3369	1.0375	2.55	0.7428	1.2629
1.00	0.3532	1.0415	2.60	0.7516	1.2727
1.05	0.3694	1.0457	2.65	0.7598	1.2831
1.10	0.3854	1.0501	2.70	0.7680	1.2934
1.15	0.4011	1.0548	2.75	0.7759	1.3039
1.20	0.4166	1.0596	2.80	0.7836	1.3146
1.25	0.4319	1.0647	2.85	0.7910	1.3254
1.30	0.4470	1.0698	2.90	0.7982	1.3367
1.35	0.4618	1.0754	2.95	0.8052	1.3478
1.40	0.4765	1.0810	3.00	0.8122	1.3589
1.45	0.4908	1.0868	3.05	0.8186	1.3708
1.50	0.5050	1.0928	3.10	0.8251	1.3821
1.55	0.5188	1.0990	3.15	0.8312	1.3942

Table 4.

with the other method for minimizing error due to the non-uniformity of the galvanometer field should be observed.

A further precaution appears in connection with the fact that the damping factor in critically damped motion is ϵ; and it follows that, unless damping is exactly critical, the damping factor will have some other value, not far from ϵ, perhaps, but of uncertain value. It is therefore important to render damping exactly critical, using the method described in connection with equation (30).

D. Measurable Intervals.

A consideration of the formulas shows that the method employing periodic motion should prove useful for timing any event, the duration of which is less than the half period of the coil. In the method of critically damped motion there is, according to the formula, no upper limit; but when the throw approaches the steady deflection in magnitude, the increase of deflection per second is small. Greater intervals than the half period of the undamped coil should, however, be accurately measurable, since when $\tau = T_0/2$, the angle of throw is only 83 percent. of the final steady deflection, as Table 4 shows. In both cases the upper limit may, of course, be extended by increasing the moment of inertia of the coil[1]; this is preferable to diminishing the torsional control, since weak control means large possible variation in the damping factor, unsteady zero, and greater probable errors from deflection hysteresis, unless special precautions are taken. When the method of critically damped motion is used, the relation between moment of inertia and controlling torque should be such that some external resistance is required to make damping critical; this minimizes error due to fluctuations of resistance of the coil with varying temperature.

The suggestion lies near at hand to employ the equations developed for calibrating the scale of a particular instrument which, let us say, is in a fixed position and has been adjusted once for all, so as to be direct reading for time. This would necessitate the use of definite resistances with the instrument, and a given voltage; and possibly a device for increasing or diminishing the effective length of the upper suspension so as to produce a slight change in the period if this should become necessary. The scale might be checked from time to time by comparison with a standard clock. The author hopes to determine the practicability of this suggestion in the near future.

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[1]Zeleny, A., Phys. Rev., O.S., 23, 404, 1906; Peirce, B. O., Am. Acad. Proc., 44, 287, 1909.

III. Experimental.

To verify the theory which leads to the equations, a Leeds and Northrup P type galvanometer having a resistance of 127.2 ohms was used. The coil was suspended by a 1.5 mil phosphor bronze strip. The connections were made as indicated in Fig. 8. G is the galvanometer and S is the shunt by means of which, upon closing the switch K_3, the instrument can be damped critically. The portion acb of the circuit serves to short-circuit the instrument; the resistance R_1 renders the resistance of acb vanishingly small compared with that of the upper branch. R_2 is a resistance of such value that the logarithmic decrement admits of applying equation (73) in which the damping factor is sufficiently accurate when written in the form $\sqrt{p}$. M and N, with the voltmeter V and the battery B_1 constitute a potentiometer arrangement permitting a current of any desired strength to be sent through the galvanometer. A and D are relays, connected through the switches K_1 and K_2 respectively to the clock circuit containing the seconds' pendulum P and the battery B_2. The relays are so adjusted as to stick after a momentary current has passed through them, thus breaking the circuit and keeping it open until set by hand for the next determination. It is noticed that the beginning and end of the interval are each marked by a break in a circuit, this being more reliable than first

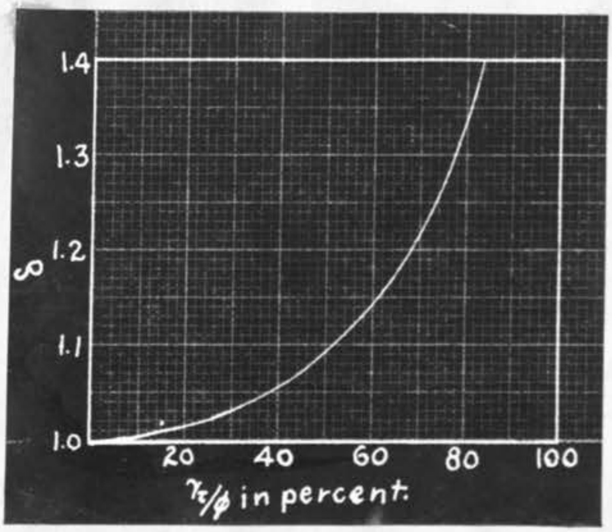
1.4
1.3
1.2
1.1
1.0
S
20
40
60
80
100
r_1/ϕ in percent.

Fig. 7.

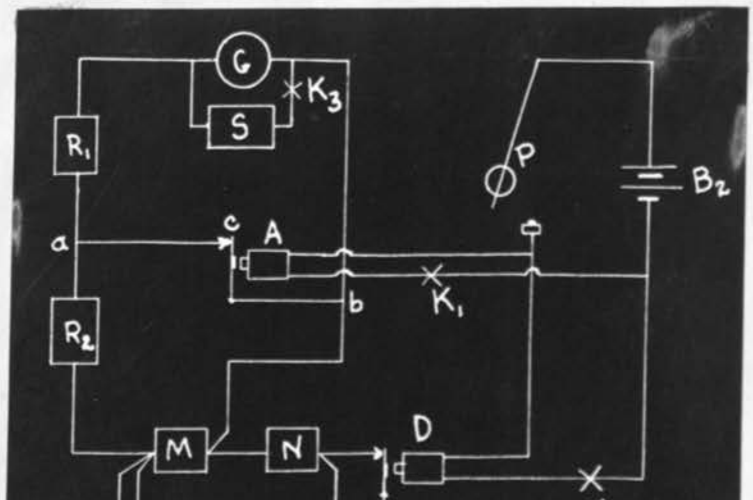
G
S
K_3
R_1
R_2
a
c
A
b
K_1
P
B_2
M
N
D
V
K_2
B_1

Fig. 8.

to "make" the circuit through the galvanometer and then to break it. The arrangement in the figure further keeps the logarithmic decrement of the coil constant whether or not a current is flowing. With these connections it is possible to keep the current flowing through the galvanometer for any desire whole number of seconds, simply by proper timing of the instants of closing the switches K_1 and K_2.

a. Results from Method of Periodic Motion.--Two representative sets of determinations to verify equation (73) are given in Table 5. There is an indication of falling off of accuracy in the sixth second, which is to be expected, inasmuch as the half period of the coil is 6.5 seconds. The manipulation is somewhat more difficult than that of using the galvanometer for the measurement of instantaneous discharges, since, for the longer intervals, throws may not be found accurately unless the coil is brought quite to rest.[1]

b. Results from Method of Critically Damped Motion.--In the fourth column of Table 5 are given results obtained in one of the sets of observation for the determination of τ by means of equation (86), with the help of Table 4. It is the writer's experience in repeated tests of both

[1]Zeleny, A., Phys. Rev., O.S., 23, 405, 1906.

(To follow page 79)

I	II	III	IV
1.000	1.002	1.000	1.001
2.000	2.004	2.004	2.000
3.000	3.009	3.000	3.001
4.000	3.998	4.068	3.994
5.000	4.996	5.008	5.000
6.000	5.968	6.038	6.006

Column I: Actual seconds by standard clock.

Columns II and III: Time determined by means of equation (73).

Column IV: Time determined by means of equation (86) and Table 4.

Table 5.

methods that this is the more satisfactory to use, both as regards easo of manipulation and simplicity of making the computation when the table of values (Table 4) is available. In fact, one may feel confident of being able to obtain a certain set of observations by this method in half the time required by the other, and that with greater accuracy, as a comparison of the results in the fourth column of Table 5 with those of the second and third columns indicates.

SUMMARY

In the foregoing paper the writer has presented the following points in connection with the theory and application of the type of galvanometer having an opan rectangular coil free to move in a magnetic field about a vertical axis, being suspended by an elastic metal fiber.

1. Development of the general theory as to the motion of such a coil under different typical conditions of damping first, when the motion is imparted to the coil impulsively while the coil is in its position of rest and second, when the motion is produced by a steady current flowing through the coil. This section of the paper may be regarded as a resumé of the theoretical work done by various workers with this instrument.

2. Relations are developed between the damping of the motion of the coil, expressed in terms of the logarithmic decrement, and the resistance in the coil circuit. In this section is given a method by which that particular resistance in the circuit may be determined which renders the motion just aperiodic.

3. Relations are developed between the damping

in the circuit

effect, the resistance in the circuit and the intensity of the field in which the coil is moving. That particular relation for which the motion of the coil remains periodic is made the basis of a method for measuring magnetic field intensities, yielding satisfactory experimental results.

4. A formula is developed for computing the size of wire to be used in a damping rectangle which is to produce any desired degree of damping when attached to the coil; particularly, to find the size of wire to be used in a damping rectangle to produce critical damping without reducing deflection sensitivity under the conditions under which the galvanometer is being used.

5. Derivations of equations for the ballistic constant of a galvanometer are given, for use under different conditions of damping; and an equation is given from which the ballistic constant on closed circuit may be determined when the constant on open circuit and the resistance of the closed circuit are known, together with the value of the logarithmic decrement corresponding to some particular resistance. Experiment shows the equations to give results dependable to 0.2 percent.

6. A method is developed for obtaining the current constant to be used in the formula for the ballistic constant by the current deflection method; and it is pointed out that non-uniformity of the magnetic field makes the usual procedure of finding the current constant inaccurate.

The results of experiment upon the modified method show it to be dependable to several tenths of a percent. accuracy; this has been further borne out by results of classes in the laboratory by whom the method was used.

7. In the next section of the paper is given the development of an equation for correcting the ballistic throws obtained on closed circuit when the discharge took place on open circuit, the closed circuit containing a thermoelectromotive force. A comparison of the results from this equation with an experimental method previously described shows good agreement between the two methods.

8. To adapt the galvanometer to the measurement of time intervals which are shorter than those which are accurately measurable by means of a stop-watch, two equations are developed. The first applies to slightly damped motion of the coil, the second to critically damped motion. In connection with the second a table is given, the use of which reduces computation to a minimum, in that it obviates a solution by approximations, which would otherwise be necessary. The indicated accuracy in time measurement by this method is 0.15 percent, and the average agreement with "standard" time intervals is still closer.

In conclusion, the author is happy to acknowledge his indebtedness ~~the~~ to the various members of the Department of Physics whose kindly interest and helpful suggestions have served to make this work a pleasure.

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